Math 511

Q's

Spanning set $\rightarrow$ all $\mathbb{V} = c_1\mathbb{V}_1 + c_2\mathbb{V}_2 + \dots + c_k\mathbb{V}_k$

same as entire vector space

ex $\boxed{\mathbb{R}^2}$ $\boxed{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}}, \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}-1\\1\end{bmatrix}$

3.2 #16 any $p(x)$ in P_3 is $p(x) = a_0 + a_1x + a_2x^2$

16) $1, x^2, x^2-2$ $\quad \mathbb{P}_1 = 1 \quad \mathbb{P}_2 = x^2 \quad \mathbb{P}_3 = x^2 - 2$

Spanning P_3 $\rightarrow$ is any $p(x)$ in P_3 capable of being rep. by $\mathbb{P}_1, \mathbb{P}_2, \mathbb{P}_3$ in linear combo?

$$a_0 + a_1x + a_2x^2 = c_1(1) + c_2(x^2) + c_3(x^2-2)$$

$$a_0 + a_1x + a_2x^2 = (c_1 - 2c_3) + (0)x + (c_2+c_3)x^2$$

no soln

given a_i $\rightarrow$ you can state c_i

3.4 $\{\mathbb{V}_1, \mathbb{V}_2, \dots, \mathbb{V}_k\}$ is minimal if

1. $\text{Span}(\{\mathbb{V}_1, \mathbb{V}_2, \dots, \mathbb{V}_k\}) = V$
2. $\mathbb{V}_i$ are linearly ind.

Def call $\mathbb{V}_i$ a $\boxed{\text{basis}}$ of V

Standard Basis

$\mathbb{R}^n$ basis $\begin{bmatrix}1\\0\\0\\ \vdots\\0\end{bmatrix}$ $\begin{bmatrix}0\\1\\0\\ \vdots\\0\end{bmatrix}$ $\begin{bmatrix}0\\0\\1\\0\\ \vdots\\0\end{bmatrix}$ $\cdots$ $\begin{bmatrix}0\\0\\0\\ \vdots\\0\\1\end{bmatrix}$

$\quad\quad\quad\quad\quad\quad e_1 \quad\quad e_2 \quad\quad e_3 \quad\quad e_n$

$\mathbb{R}^3 \to \begin{bmatrix}1\\0\\0\end{bmatrix} \begin{bmatrix}0\\1\\0\end{bmatrix} \begin{bmatrix}0\\0\\1\end{bmatrix} \to \begin{bmatrix}a\\b\\c\end{bmatrix} = a\begin{bmatrix}1\\0\\0\end{bmatrix} + b\begin{bmatrix}0\\1\\0\end{bmatrix} + c\begin{bmatrix}0\\0\\1\end{bmatrix}$

$P_n \to 1, x, x^2, \ldots, x^{n-1}$

$\mathbb{R}^{2\times 2} \to \begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\end{bmatrix}$

Th^m if $\{v_1, v_2, \ldots, v_n\}$ is a spanning set of V

$\to$ any m-vectors ($m > n$) are lin dep.

Corollary if $v_1, v_2, \ldots, v_n$ and $u_1, u_2, \ldots, u_m$ are basis of V.

$\to \quad n = m$ (smallest # of spanning vectors is uniq)

Def $\dim(V) = n$ where n-vectors form a basis of V.

Special Cases ① $\dim(\{0\}) = 0$

② $\dim(V) = n \in \mathbb{N}$ V is called finite dimension

③ if not .. infinite dimension.

ex $P = P_\infty$ (set of all polynomials)

$\dim(P) = \infty$

pf (by contradiction)

assume $\dim(P) = n$

$\rightarrow$ $1, x, x^2, \dots, x^{n-1}$ are my standard basis

$\rightarrow$ $1, x, x^2, \dots, x^n$ must be <u>dep.</u>

$$\begin{vmatrix} 1 & x & x^2 & \cdots & x^n \\ 0 & 1 & 2x & \cdots & nx^{n-1} \\ 0 & 0 & 2 & \cdots & n(n-1)x^{n-2} \\ \vdots & 0 & & \ddots & \\ & & & & n! \end{vmatrix} = \text{prod of diagonal} \neq 0$$

$\rightarrow$ $1, x, \dots, x^n$ are <u>independent.</u>

Contradiction so $\dim(P) = \infty$

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thm $\dim(V) = n > 0$

① any set of n lin. ind. vectors will span V.

② any n vectors that span V are lin. ind.

$\dim(V) = n$

→ you have $< n$ vectors?

→ you have $> n$ vectors?

$\boxed{\text{th}^{m}}$ $\dim(V) = n > 0$

① no fewer than n-vectors can span V.

② (fewer than n?)

we can $\boxed{\text{extend}}$ our vectors by adding new vectors that are lin. ind. to what we have until we have a basis of n-vectors

$\boxed{\text{ex}}$ $\mathbb{R}^3$ $u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $u_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

add? $\cancel{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}$ b/c its dep.

add? $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ check ind.

$$c_1 u_1 + c_2 u_2 + c_3 u_3 = 0$$

$$\begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 1 \neq 0$$ so yes ind.

② (more than n?)

we $\boxed{\text{pare down}}$ the vectors by removing dep. vectors.

$\mathbb{R}^3$

$$\underbrace{\begin{bmatrix}1\\0\\0\end{bmatrix} \begin{bmatrix}0\\1\\0\end{bmatrix}}_{\text{D(+) one more (extend) to make a basis for } \mathbb{R}^3} \underbrace{\cancel{\begin{bmatrix}1\\1\\0\end{bmatrix}} \cancel{\begin{bmatrix}2\\0\\0\end{bmatrix}} \cancel{\begin{bmatrix}0\\\pi\\0\end{bmatrix}}}_{\text{pare down}}$$

$\boxed{\text{Thm } 3.3.8}$

given $\mathbb{V} = c_1 \mathbb{b}_1 + c_2 \mathbb{b}_2 + \dots + c_k \mathbb{b}_k$

and $\mathbb{b}_1, \mathbb{b}_2, \dots, \mathbb{b}_k$ are a basis of V

$\dim(V) = K$

$\rightarrow$ $\boxed{c_i \text{ are uniq.}}$

$\boxed{\text{So}}$

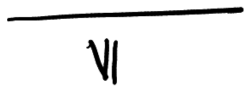
we can represent $\mathbb{V}$ using the uniq c_i's

$\begin{bmatrix}c_1\\c_2\\\vdots\\c_k\end{bmatrix}$ are $\mathbb{V}$'s coordinates in $\mathbb{b}_i$ basis

Notation: $[c]_{\mathbb{b}}$

coordinates basis label

ex



$\mathbf{v} = [1]_{\text{feet}}$

$\mathbf{v} = [12]_{\text{inch}}$

→ <u>change of basis?</u>

$[12]_{in} = \mathbf{v} = [1]_{ft}$

$12\,in = 1\,ft$

$$2.1\,\cancel{ft}\left(\frac{12\,in}{1\,\cancel{ft}}\right) = \underline{\underline{25.2\,in}} \qquad \frac{12\,in}{1\,ft} = 1$$

$2 + \frac{1}{10}$

Vectors:

basis #1 $\quad B = [\mathbf{b}_1\ \mathbf{b}_2 \cdots \mathbf{b}_k]$

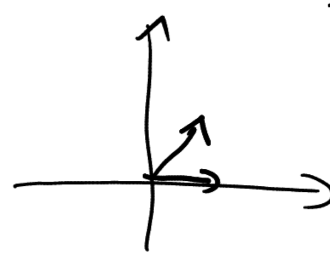
basis #2 $\quad D = [\mathbf{d}_1\ \mathbf{d}_2 \cdots \mathbf{d}_k]$

(1) $B[\mathbf{c}]_B \qquad \mathbf{v} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_k\mathbf{b}_k$

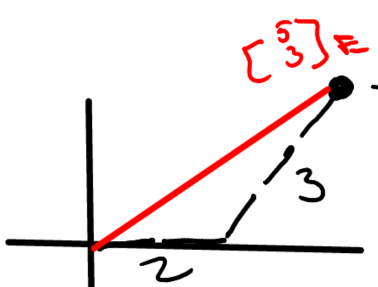
$$[\mathbf{b}_1\ \mathbf{b}_2 \cdots \mathbf{b}_k]\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} = \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_k \end{bmatrix}_E$$

$\mathbb{R}$ in <u>standard basis</u>

ex $\mathbb{R}^2 \qquad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$



ex



$\to \begin{bmatrix} 2 \\ 3 \end{bmatrix}_B$

use $\quad B[\mathbf{c}]_B$ = standard basis $\mathbf{v}$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix}_B = \begin{bmatrix} 5 \\ 3 \end{bmatrix}_E$$

(1) Convert from base B to standard.

$$B\,[a]_B = [a]_E$$

(2) Convert from standard to base B?

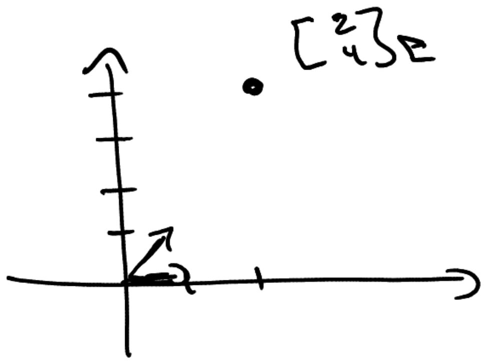
$$[a]_B = B^{-1}[a]_E$$

$\boxed{\text{ex}}$ $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

B^{-1}?

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \end{array}\right]$$

B^{-1}



$$[a]_B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix}_E$$

$$= \begin{bmatrix} -2 \\ 4 \end{bmatrix}_B$$

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(3) Convert from base B to/from base D

$$B[a]_B = [a]_E$$
$$\parallel$$
$$D[a]_D = [a]_E$$

$\rightarrow$ $\underline{B[a]_B = D[a]_D}$

a) given $[a]_D \rightarrow$ get $[a]_B$

$$[a]_B = \underbrace{B^{-1}D}_{=S}[a]_D \quad ✓$$

b) given $[a]_B \rightarrow$ get $[a]_D$

$$[a]_D = \underbrace{D^{-1}B}_{=S^{-1}}[a]_B \quad ✓$$

ex $\mathbb{R}^3$ basis $W = \begin{bmatrix} 3 & 0 & 0 \\ 2 & 4 & 0 \\ 1 & 5 & 6 \end{bmatrix}$

$$Z = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}_W \rightarrow \begin{bmatrix} ? \\ ? \end{bmatrix}_Z$ use $W[c]_W = Z[c]_Z$

$$Z^{-1} W [c]_W = [c]_Z$$

$$\boxed{\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \underbrace{\begin{bmatrix} 3 & 0 & 0 \\ 2 & 4 & 0 \\ 1 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}_W} = \begin{bmatrix} \; \end{bmatrix}_Z}$$

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ -6 \\ -9 \end{bmatrix} \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ 0 & 1 & 3 & -6 \\ 0 & 0 & 1 & -9 \end{array}\right]$$

$A^{-1} d \rightarrow [A \mid d]$

use row ops $[I \mid A^{-1} d]$

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 21 \\ 0 & 1 & 0 & 21 \\ 0 & 0 & 1 & -9 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 42 \\ 0 & 1 & 0 & 21 \\ 0 & 0 & 1 & -9 \end{array}\right]$$

$$\begin{bmatrix} 42 \\ 21 \\ -9 \end{bmatrix}_Z$$

Why other Basis if we have standard?

ex

Markov Chain / Markov Process

Inductive basis $\mathbf{x}_0$ Inductive Rule $\mathbf{x}_n = A\mathbf{x}_{n-1}$

seq $\mathbf{x}_0,\ \mathbf{x}_1 = A\mathbf{x}_0,\ \mathbf{x}_2 = A\mathbf{x}_1,\ \mathbf{x}_3 = A\mathbf{x}_2, \ldots$

$$x_n = A^n x_0$$

ex City suburb

	city	suburb
city	94%	2%
suburb	6%	98%

$\rightarrow \begin{bmatrix} .94 & .02 \\ .06 & .98 \end{bmatrix} = A$

this year $\begin{bmatrix} x_{city} \\ x_{sub.} \end{bmatrix} = \mathbf{x}$ $\begin{bmatrix} .94 & .02 \\ .06 & .98 \end{bmatrix}\begin{bmatrix} x_{city} \\ x_{sub} \end{bmatrix} = \begin{matrix} \text{New city} \\ \text{new sub} \end{matrix}$

$$\mathbf{x}_0,\ \underline{A\mathbf{x}_0 = \mathbf{x}_1},\ A^2\mathbf{x}_0 = \mathbf{x}_2, \ldots\ \underline{A^n x_0 = \mathbf{x}_n}$$

Notice about A. has two special vectors.

$$A\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \underline{.92}\begin{bmatrix} -1 \\ 1 \end{bmatrix} \leftarrow \mathbf{v}_1$$

$$A\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \underline{1}\begin{bmatrix} 1 \\ 3 \end{bmatrix} \leftarrow \mathbf{v}_2$$

so for any $\mathbf{x}_0 = \underline{c_1\mathbf{v}_1 + c_2\mathbf{v}_2}$ use coord. of these $\mathbf{v}_{1,2}$

$$\mathbf{x}_1 = A\mathbf{x}_0 = A(c_1\mathbf{v}_1 + c_2\mathbf{v}_2)$$

$$= c_1 A\mathbf{v}_1 + c_2 A\mathbf{v}_2$$

$$= c_1(.92)\mathbf{v}_1 + c_2\mathbf{v}_2$$

$$\mathbf{x}_n = A^n\mathbf{x}_0 = c_1\cancel{(.92)^n}\mathbf{v}_1 + \boxed{c_2(1)^n\mathbf{v}_2}$$

$\to 0$
