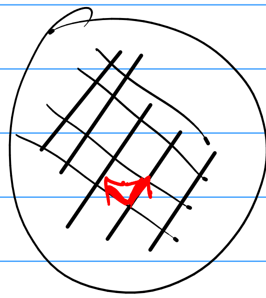
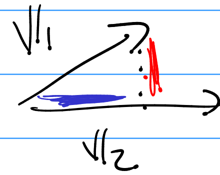
Math 511

Ch 5 Vector Spaces → basis
→ standard basis

ex) $\mathbb{R}^2$

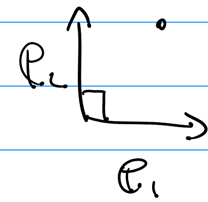


$\alpha v_1 + \beta v_2 = v$



vs

$v = a e_1 + b e_2$ $\begin{bmatrix} a \\ b \end{bmatrix}_E$



Any Vector Space ..

$\mathbb{R}^2/\mathbb{R}^3$ ① length? → idea of it : Norm

$\mathbb{R}^2/\mathbb{R}^3$ ② angle? → idea that covers this : Inner Product

Study $\mathbb{R}^2$ / $\mathbb{R}^3$

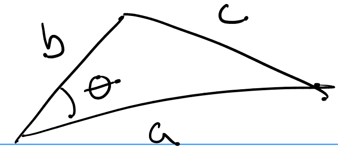
5.1 Scalar Product in $\mathbb{R}^n$

def: $\boxed{x^T y = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n}$

① length $\|x\| = (x^T x)^{1/2}$

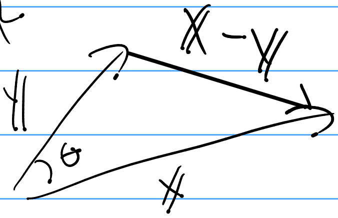
$\mathbb{R}^2$ $\|x\| = \boxed{\sqrt{x_1^2 + x_2^2}}$ $\mathbb{R}^3 \Rightarrow \|x\| = \boxed{\sqrt{x_1^2 + x_2^2 + x_3^2}}$

(2)



$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

$\mathbb{R}^2 \,|\, \mathbb{R}^3$



$$\|\underline{x - y}\|^2 = \|x\|^2 + \|y\|^2 - 2\|x\|\,\|y\|\cos\theta$$

$$(x - y)^T (x - y) = x^T x + y^T y - 2\,\|x\|\,\|y\|\cos\theta$$

$$\boxed{x^T y = \|x\|\,\|y\|\cos\theta}$$

$$\boxed{\cos\theta = \frac{x^T y}{\|x\|\,\|y\|}} = \frac{x^T y}{\sqrt{x^T x}\,\sqrt{y^T y}}$$

So $\mathbb{R}^2 \,|\, \mathbb{R}^3$

(1) length $\quad \|x\| = \sqrt{x^T x}$

(2) angle $\quad \cos\theta = \dfrac{x^T y}{\|x\|\,\|y\|}$

So $x^T y$ the scalar product is a metric that studys the idea of <u>length, angle</u>

Note:

$$x^T y = \|x\| \, \|y\| \cos\theta$$

$$\cos\theta = \frac{x^T y}{\|x\| \|y\|}$$

Th^n Cachy - Schwarz Ineq.

$$|x^T y| \leq \|x\| \, \|y\|$$

Now

$$x^T y = \|x\| \, \|y\| \cos\theta$$

Q you find that $x^T y = 0$

$$0 = \|x\| \cdot \|y\| \cdot \cos\theta$$

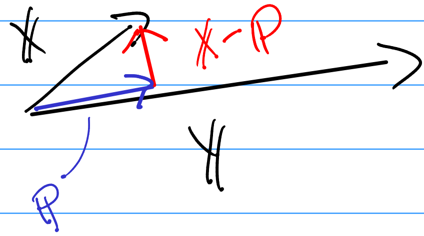
$\rightarrow$ $x = 0$ or $y = 0$ or $\cos\theta = 0$

if you state that 0 is always orthogonal to any vector.

Def in $\mathbb{R}^2 / \mathbb{R}^3$ x, y are called orthogonal if $x^T y = 0$

$\rightarrow$ we have a metric for $\|x\|$ and $x \perp y$

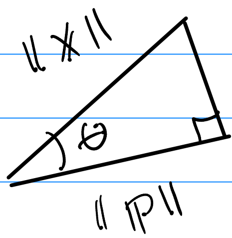
Q what about non-orthogonal?



$$x^T y \neq 0$$

$\rightarrow$ can we find p?

how?



$$\cos\theta = \frac{\|p\|}{\|x\|} \rightarrow \|p\| = \|x\| \cos\theta$$

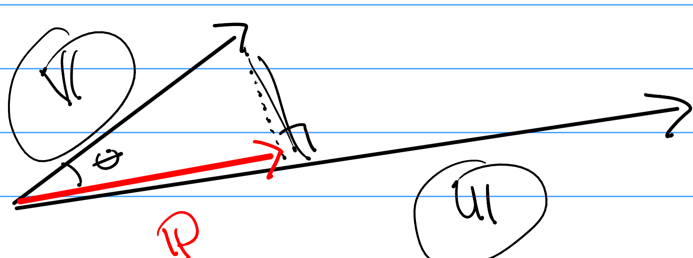
$$\rightarrow \|p\| = \cancel{\|x\|} \frac{x^T y}{\cancel{\|x\|}\|y\|} = \boxed{\frac{x^T y}{\|y\|}}$$

① **Def:** $\|p\| = \alpha = \dfrac{x^T y}{\|y\|}$ called [scalar projection]

② **Def:** $p = \alpha \dfrac{y}{\|y\|} = \dfrac{x^T y}{\|y\|} \dfrac{y}{\|y\|} = \boxed{\dfrac{x^T y}{y^T y} y}$

is called the [vector projection]

ex

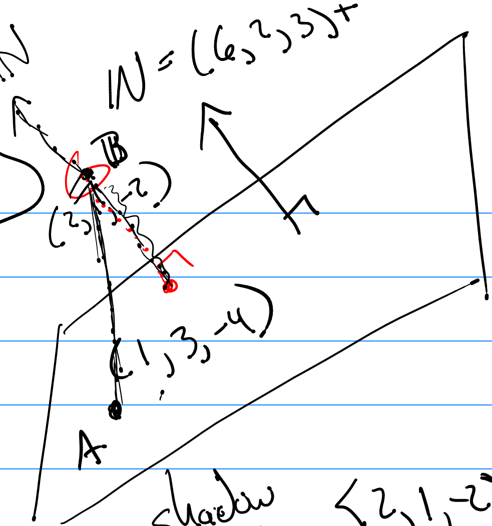


project v onto u

$$\boxed{\begin{aligned} \|p\| &= \frac{v^T u}{\|u\|} \\ p &= \boxed{\frac{v^T u}{u^T u}} u \end{aligned}}$$

★

ex



plane in $\mathbb{R}^3$

$$6(x-1)+2(y-3)+3(z+4)=0$$

shadow of $\langle 2,1,-2\rangle$ to $\langle 1,3,-4\rangle$ onto $\mathbb{N}$ would be distance.

vector

$$\vec{AB} = \mathbf{v} = (1,-2,2)^T$$

$$d = \frac{(1,-2,2)^T(6,2,3)}{\|(6,2,3)\|} = \boxed{\frac{8}{7}}$$

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use $x^T y$ in $\mathbb{R}^2 \mid \mathbb{R}^3$ for "angles", "length"

→ $\mathbb{R}^n$? it ends up that

$$x^T y = x_1 y_1 + x_2 y_2 + \dots + x_n y_n$$

still works!

(1) $\|x\| = (x^T x)^{1/2}$

(2) show $|x^T y| \le \|x\|\,\|y\|$

$$\to \quad -1 \le \frac{\mathbb{x}^T \mathbb{y}}{\|\mathbb{x}\| \|\mathbb{y}\|} \le 1$$

define: $$\boxed{\cos\theta = \frac{\mathbb{x}^T \mathbb{y}}{\|\mathbb{x}\| \|\mathbb{y}\|}} \qquad 0 \le \theta \le \pi$$

③ $\boxed{\mathbb{x} \perp \mathbb{y} \text{ if } \mathbb{x}^T \mathbb{y} = 0}$ orthogonal

④ Nice bonus

$$\|\mathbb{x}+\mathbb{y}\|^2 = (\mathbb{x}+\mathbb{y})^T(\mathbb{x}+\mathbb{y})$$

$$= \mathbb{x}^T\mathbb{x} + 2\mathbb{x}^T\mathbb{y} + \mathbb{y}^T\mathbb{y}$$

So $\|\mathbb{x}+\mathbb{y}\|^2 = \|\mathbb{x}\|^2 + 2\mathbb{x}^T\mathbb{y} + \|\mathbb{y}\|^2$

if $\mathbb{x} \perp \mathbb{y} \to \boxed{\mathbb{x}^T\mathbb{y}} = 0$

Pythagorean Law

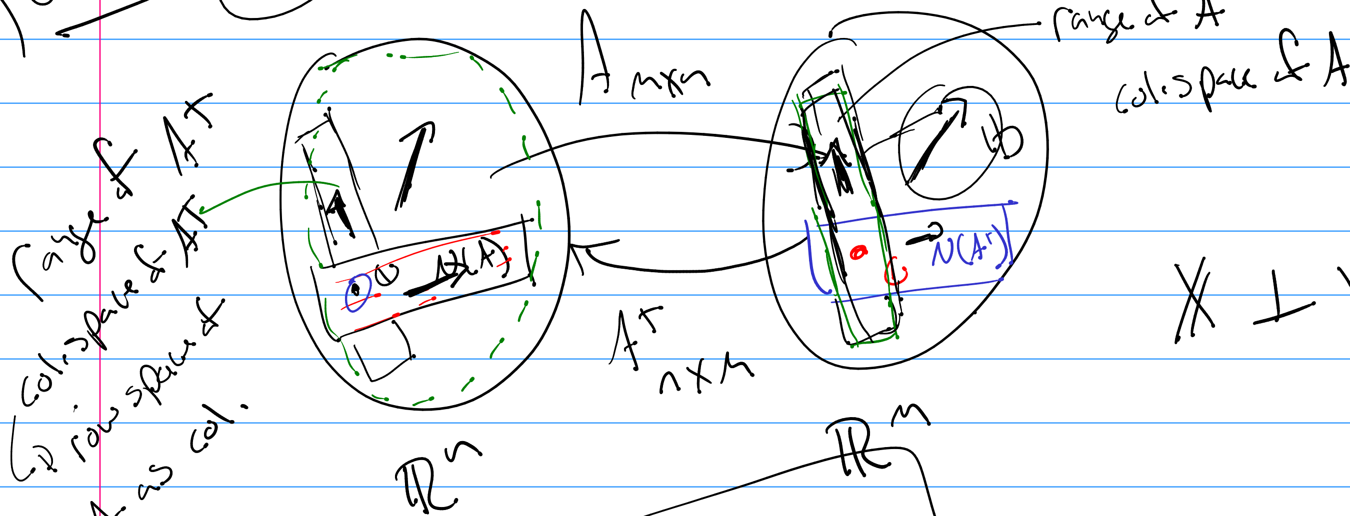
So $\boxed{\text{if } \mathbb{x} \perp \mathbb{y} \qquad \|\mathbb{x}+\mathbb{y}\|^2 = \|\mathbb{x}\|^2 + \|\mathbb{y}\|^2}$

$\boxed{\text{Ch 5}}$ 5.1 – 5.3 is $\|x\|$, $x \perp y$ in $\mathbb{R}^n$

with applications.

5.4 → onto $\underline{\text{any}}$ V, vector space

$\boxed{\text{Goal}}$ $\mathbb{R}^n$



$X \perp Y$

$$\boxed{Ax = b}$$

$$A^T \left(b = \alpha \cancel{N(A^T)} + \beta\, \boxed{R(A)} \right)$$

$$\underset{m\times n}{A} x = b \leftarrow$$

$$\to \boxed{\underbrace{A^T A}_{\substack{n\times m\ m\times n \\ n\times n}} x = A^T b} \leftarrow$$