Math 511

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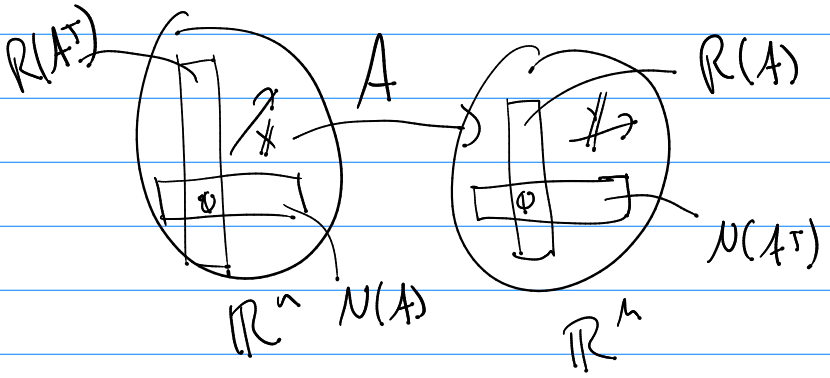
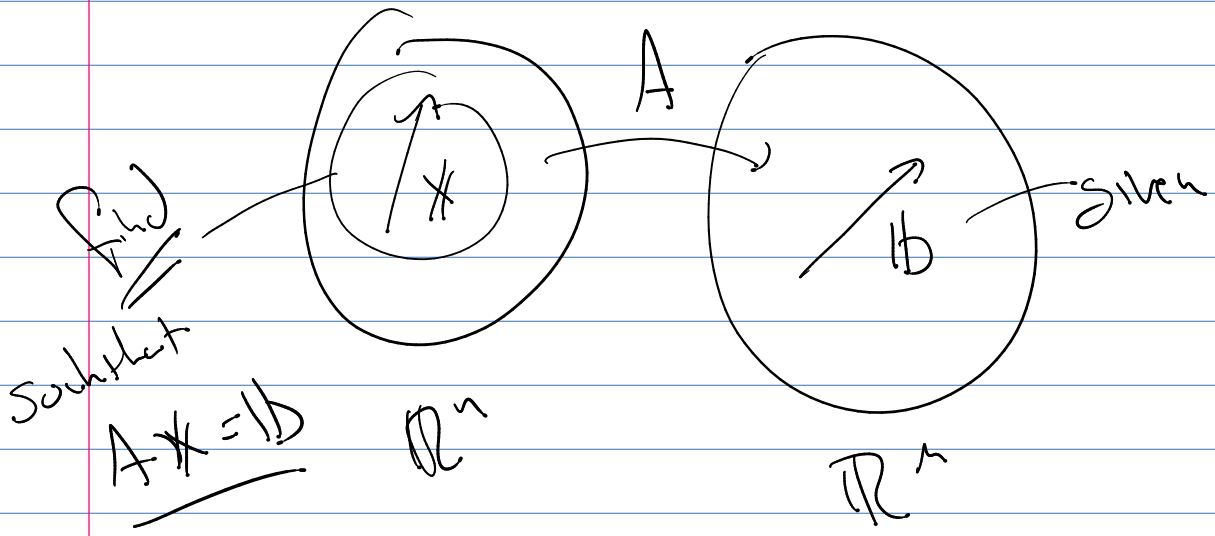
$A_{m \times n}$



54 systems & linear equations: Solve $A\vec{x} = \vec{b}$



① ∞ soln prob. $A\vec{x} = \vec{b} \rightarrow [A \mid \vec{b}]$

$\xrightarrow{RREF}$ $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

② 1 soln. $A\vec{x} = \vec{b}$

$\rightarrow \vec{x} = A^{-1}\vec{b}$

A^{-1} exists, $\det(A) \neq 0$, Square.

③ $Ax = b$ has no solns

typically overdetermined

$Ax = b \rightarrow [A|b]$

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{array}\right] \leftarrow \text{no soln}$$

Ex

7 pts. $y = ax^2 + bx + c$

that goes through them?

$X_{7\times 3}$

(x_1, y_1) $ax_1^2 + bx_1 + c = y_1$

(x_2, y_2) $ax_2^2 + bx_2 + c = y_2$ $\Rightarrow \begin{bmatrix} x^2 & x^1 & x^0 \end{bmatrix} C = y$

(x_7, y_7) $ax_7^2 + bx_7 + c = y_7$

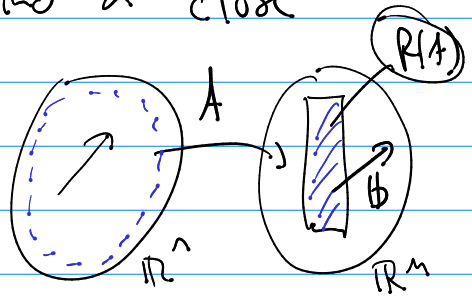
x is x-coord y is y coord C as $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$

has no soln but can I get "close"?

$Ax = b$ has no soln. can I find a 'close' soln?

For all $x \in \mathbb{R}^n$

Defn: $r(x) = b - Ax$ is the residual



Find smallest $\|r(x)\| = \sqrt{r(x)^T r(x)}$

rather use $\|r(x)\|^2$ to make problem easier.

So we have turned "find closest Ax to b"

into: minimize $\|r(x)\|^2 = \|b - Ax\|^2$

New Prob: find $x \in \mathbb{R}^n$ such that $\|b - Ax\|^2$ is Minimum. (least squares problem)

if you find a soln, call it $\hat{x}$ and

$\hat{x}$ is the soln to the least squares problem

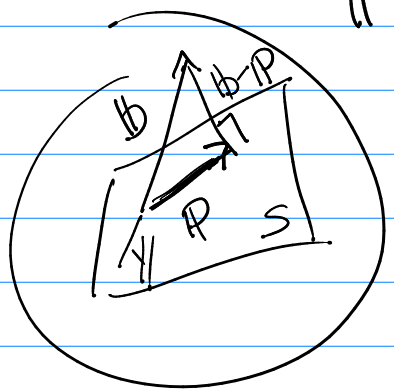
Thm: S is a subspace of $\mathbb{R}^m$ for each $b \in \mathbb{R}^m$

$\exists!$ $p \in S$ such that it is closest to b

(means) $\forall y \in S$ and $y \neq p$

$$\|b - p\| \leq \|b - y\|$$

Visual:



Find: importantly

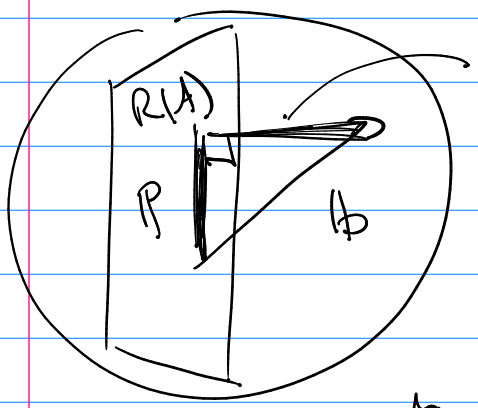
$(b - p) \in S^\perp$

we know $R(A)$ is a subspace of $\mathbb{R}^m$

So to find minimize $\| b - \underbrace{Ax}_{\text{in } R(A)} \|^2$

by thm we have a closest $p \in R(A)$ and

$$(b - p) \in R(A)^\perp = N(A^T)$$



$b - p \in N(A^T)$

so there is $\hat{x} \in \mathbb{R}^n$

such that $A\hat{x} = p$

↑ least squares soln.

① $Ax = b$ has no soln.

② minimize $\| b - Ax \|^2$

③ find $\hat{x}$ such that $A\hat{x} = p \in R(A)$

where $p \in R(A)$ is closest to b.

Note: b/c $b - p \in N(A^T) \rightarrow b - A\hat{x} \in N(A^T)$

Recall: $A^T(b - A\hat{x}) = 0 \rightarrow A^T b - A^T A\hat{x} = 0$

$$\rightarrow A^T A \hat{x} = A^T b$$

Ver #4 Solve $A^TA\hat{x} = A^Tb$

Note: $A\hat{x} = b$ has no soln.

Now to solve $A^TA\hat{x} = A^Tb$ if it has a soln it will be the least squares soln.

Thm if $A_{m\times n}$ is $\text{rank}(A) = n$

$\rightarrow A^TA\hat{x} = A^Tb$ has a unq. soln.

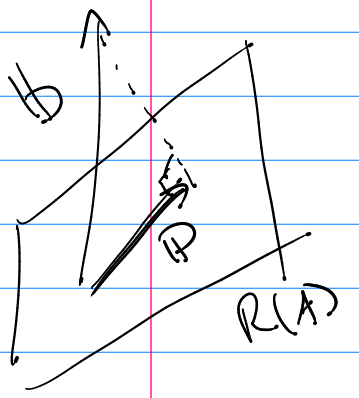
(we call it the unq least squares soln to $A\hat{x} = b$.) $A\hat{x} = b$

Soln find $\hat{x}$ to solve $\boxed{A^TA\hat{x} = A^Tb}$ A

$\rightarrow \boxed{\hat{x} = (A^TA)^{-1}(A^Tb)}$

$\rightarrow p = A\hat{x} = \boxed{A(A^TA)^{-1}A^T}b$

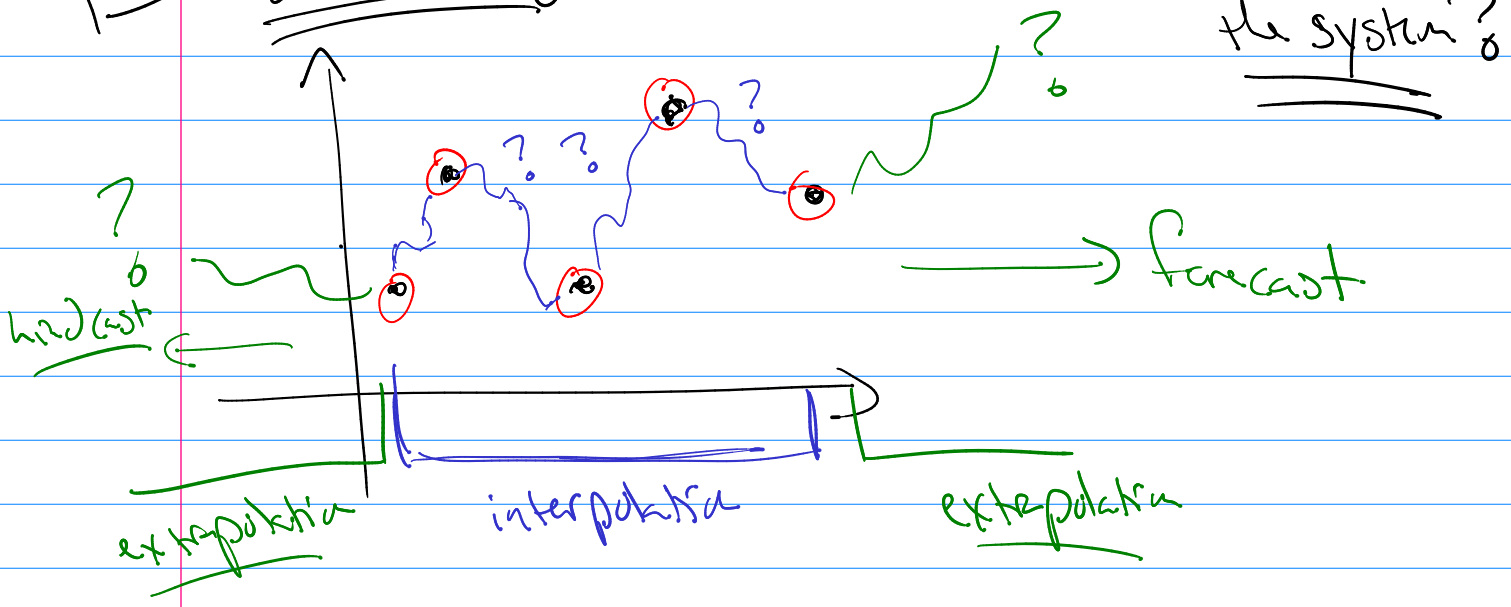
P : projection matrix

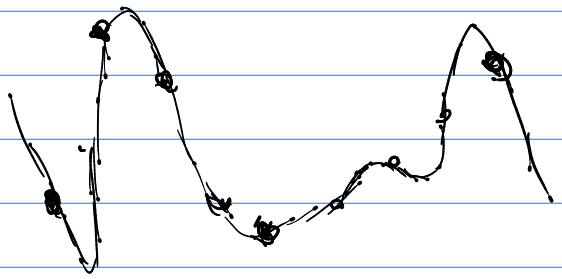


given $P = A(A^TA)^{-1}A^T$

then $p = Pb$

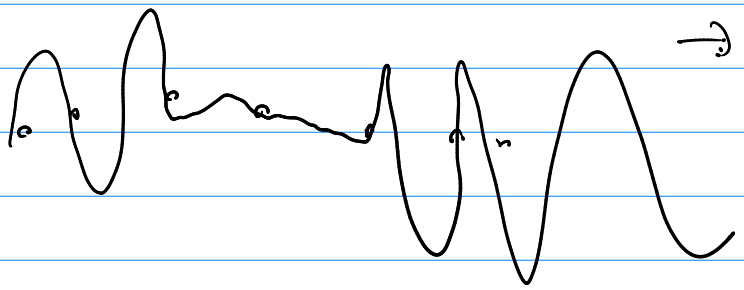
ex's Data Fitting → find a function that models the system?





4 pts → $y = a_1 + a_2x + a_3x^2 + a_4x^3$

↳ 4 term poly is exactly soln.



→ interpolating polynomial

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ex Least square poly. fits

Given K-points fit with poly < K-terms.

ex3 deg 4 poly & 10 points.

$y = ax^4 + bx^3 + cx^2 + dx + e$

→ Points: x_1, y_1
x_2, y_2
x_{10}, y_{10}

let $\mathbb{X} = [x_1\ x_2 \dots x_{10}]^T$

$\mathbb{Y} = [y_1\ y_2 \dots y_{10}]^T$

Solve: $\underbrace{[\mathbb{X}^4\ \mathbb{X}^3\ \mathbb{X}^2\ \mathbb{X}^1\ \mathbb{X}^0]}_{\text{let } \chi = A} \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \mathbb{Y}$

Solve for least squares soln.

$$\boxed{\chi^T \chi\, c = \chi^T \mathbb{Y}} \quad A$$