Math 511

$a_1, a_2, \ldots, a_n \to q_1\ q_2 \cdots q_n$ (orthonormal)

Basis $\boxed{q_1 = \frac{1}{\|a_1\|} a_1}$

Inductive step | Know: $\boxed{q_1, \ldots, q_k}$

to get q_{k+1}, $\boxed{a_k}$ − (projection of a_k onto span$(q_1, \ldots q_k)$) $= \mathbb{P}$

$$q_{k+1} = \frac{1}{\|a_k - \mathbb{P}\|}(a_k - \mathbb{P})$$

(ex)

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} = A = QR$$

$$Q = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/2 \\ 0 & 1/\sqrt{2} & -1/2 \\ 1/\sqrt{2} & 0 & -1/2 \\ 0 & 1/\sqrt{2} & 1 \end{bmatrix}$$

$$R = \begin{bmatrix} \sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & \sqrt{2} & 1/\sqrt{2} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Basis: $q_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

$\boxed{\text{Inductive:}}$ q_2 is $(a_2 - \underbrace{\text{proj. onto } q_1}_{\mathbb{P}_1}) / (\text{length})$

$\mathbb{P}_1 = \boxed{\langle a_2, q_1 \rangle q_1} = 0 \cdot q_1 = 0$

$$q_2 = \frac{1}{\sqrt{2}}\left[\begin{bmatrix}0\\1\\0\\1\end{bmatrix} - \begin{bmatrix}0\\0\\0\\0\end{bmatrix}\right] = \begin{bmatrix}0\\1/\sqrt{2}\\0\\1/\sqrt{2}\end{bmatrix}$$

$a_2 - P_1$

q_3 is $(a_3 - \underbrace{\text{proj. of } a_3 \text{ onto } q_1, q_2}_{P_2}) / (\text{length})$

$$P_2 = \boxed{\underbrace{\langle a_3, q_1\rangle}_{1/\sqrt{2}} q_1} + \underbrace{\langle a_3, q_2\rangle}_{1/\sqrt{2}} q_2 = \begin{bmatrix}1/2\\1/2\\1/2\\1/2\end{bmatrix}$$

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1/\sqrt{2}\\0\\1/\sqrt{2}\\0\end{bmatrix} + \frac{1}{\sqrt{2}}\begin{bmatrix}0\\1/\sqrt{2}\\0\\1/\sqrt{2}\end{bmatrix}$$

$$q_3 = \underbrace{\begin{bmatrix}1\\0\\0\\1\end{bmatrix} - \begin{bmatrix}1/2\\1/2\\1/2\\1/2\end{bmatrix}}_{a_3 - P_2} \Bigg/ \underbrace{\sqrt{\tfrac{1}{2}^2 + \tfrac{1}{2}^2 + \tfrac{1}{2}^2 + \tfrac{1}{2}^2}}_{1} = \begin{bmatrix}1/2\\-1/2\\-1/2\\1/2\end{bmatrix}$$

Gram-Schmidt is computationally (by computer) unstable (error blows up the problem)

→ Modified Gram-Schmidt

$$q_1 = \frac{1}{\|a_1\|} a_1$$

how much a_2 in q_1

$$q_2 = (a_2 - \boxed{}) / (\text{length})$$

how much a_3 in q_1 — how much a_3 in q_2

$$q_3 = (a_3 - \boxed{} - \boxed{}) / (\text{length})$$

$$q_4 = \left(a_4 - \overset{\text{in } q_1?}{\boxed{\quad}} - \overset{\text{in } q_2?}{\boxed{\quad}} - \overset{\text{in } q_3?}{\boxed{\quad}}\right) / \left(\| y \|\right)$$

loop on $k = 1, 2, \dots, n$

$r_{kk} = \| a_k \|$

$q_k = \dfrac{1}{r_{kk}} a_k$

loop $j = k+1, k+2, \dots, n$

$r_{kj} = \langle a_j, q_k \rangle$

$a_j = a_j - r_{kj} q_k$

end

end

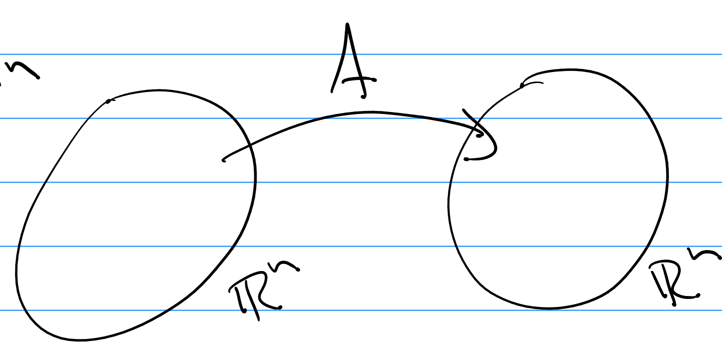
$\boxed{\text{To Do}}$ Read 5.7

q_i are orthonormal

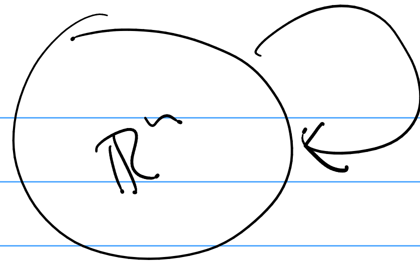
$\mathbb{P} = \langle v, q_1 \rangle q_1 + \langle v, q_2 \rangle q_2 + \dots + \langle v, q_n \rangle q_n$

$\mathbb{P} \in \operatorname{span}(q_i)$ closest to v.

$\boxed{\text{New Idea}}$ A is $n \times n$

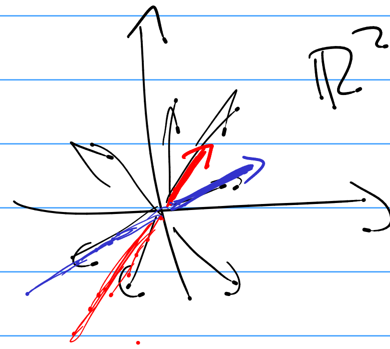


Visualize



$A_{n \times n}$

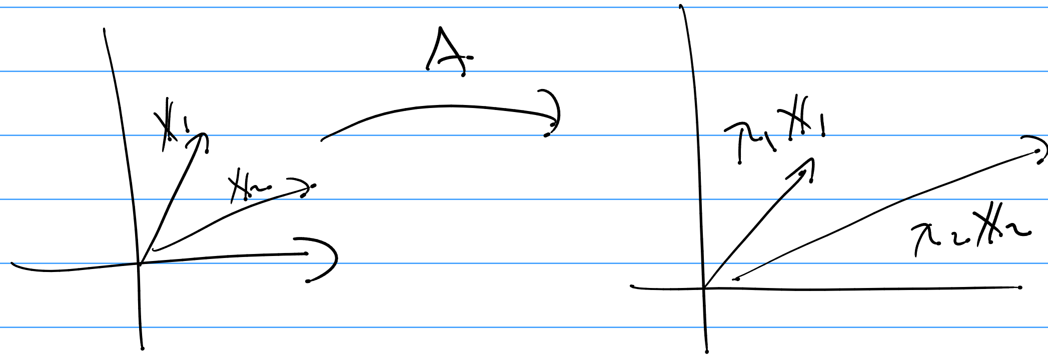
Idea: $A = \begin{bmatrix} 6 & -4 \\ 3 & -1 \end{bmatrix}$



$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$

$$A x_1 = x_2$$

if you notice $A x = \lambda x$ for some special directions.



call λ_i: eigen values

x_i: the associated eigen vectors to λ_i

→ to find: solve $A x = \lambda x$

Def for $A_{n \times n}$, λ is a eigen value (or characteristic value) of A if $\exists\, x \neq 0$ such that $A x = \lambda x$

→ x is the eigen vector (or characteristic vector) belonging to λ.

why belong to λ?

b/c if $A\mathbf{x} = \lambda \mathbf{x}$

$\rightarrow$ consider $\alpha \mathbf{x}$

$$A(\alpha \mathbf{x}) = \lambda(\alpha \mathbf{x})$$
$$\cancel{\alpha} A \mathbf{x} = \cancel{\alpha} \lambda \mathbf{x}$$
$$A\mathbf{x} = \lambda \mathbf{x}$$

$\rightarrow$ for every λ you don't get a specific vector you get a span.

Solve $A\mathbf{x} = \lambda \mathbf{x}$ $\rightarrow$ find all λ and their corresponding vectors (span)

$\rightarrow$ $(A - \lambda I)\mathbf{x} = \mathbf{0}$

all equiv. statements: ① λ is an eigen value

② $(A - \lambda I)\mathbf{x} = \mathbf{0}$ has a non-triv. soln.

③ $N(A - \lambda I) \neq \{\mathbf{0}\}$

④ $A - \lambda I$ is singular

⑤ $\det(A - \lambda I) = 0$

ex $A = \begin{bmatrix} 6 & -4 \\ 3 & -1 \end{bmatrix}$ find λ

Solve $\begin{vmatrix} 6-\lambda & -4 \\ 3 & -1-\lambda \end{vmatrix} = 0$

$(6-\lambda)(-1-\lambda) - (-12) = 0$

$\boxed{\lambda^2 - 5\lambda + 6} = 0$

this n^{th} degree poly $\equiv$ characteristic polynomial

$(\lambda - 2)(\lambda - 3) = 0$

$\boxed{\lambda = 2}$ $\boxed{\lambda = 3}$ eigen values of A.

So $|A - \lambda I| = 0$

becomes $p(\lambda) = 0$ $p(\lambda)$ is a n^{th} degree poly

Solve to get $\lambda_1, \lambda_2, \ldots, \lambda_n$

Now given a λ_i can we find corresponding eigen vector?

Remember: $(A - \lambda I)\,\mathbb{X} = \mathbb{O}$

for a specific λ_i

$\rightarrow (A - \lambda_i I)\,\mathbb{X} = \mathbb{O}$

ex $\quad A = \begin{bmatrix} 6 & -4 \\ 3 & -1 \end{bmatrix} \qquad \lambda_1 = 2$

$\Rightarrow \begin{bmatrix} 6-2 & -4 \\ 3 & -1-2 \end{bmatrix} \mathbb{X} = \mathbb{O}$

Solve for $\mathbb{X}$

$\Rightarrow$ to find $N(A - \lambda_i I)$

$\begin{bmatrix} 4 & -4 \\ 3 & -3 \end{bmatrix} \mathbb{X} = \mathbb{O}$

$\left[\begin{array}{cc|c} 4 & -4 & 0 \\ 3 & -3 & 0 \end{array}\right]$

$\left[\begin{array}{cc|c} \boxed{1} & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow \mathbb{X} = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$x_2 = \alpha$

$x_1 = \alpha$

$\lambda_1 = 2$ has corresp. $\mathbb{X}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2 = 3$

Solve $\begin{bmatrix} 6-3 & -4 \\ 3 & -1-3 \end{bmatrix} \mathbb{X} = 0$

$\begin{bmatrix} 3 & -4 \\ 3 & -4 \end{bmatrix} \mathbb{X} = 0 \rightarrow \left[\begin{array}{cc|c} 3 & -4 & 0 \\ 3 & -4 & 0 \end{array}\right]$

$\rightarrow \left[\begin{array}{cc|c} 3 & -4 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $\mathbb{X} = \alpha \begin{bmatrix} 4/3 \\ 1 \end{bmatrix}$

$x_2 = \alpha$ $x_1 = 4/3\,\alpha$

$\lambda_2 = 3$ has corr. $\mathbb{X} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$