Math 511

Q's/ Linear Normed Space

→ V is a vector space
→ function that is a metric for "length" ; Norm

5.4 #13 ① Vector Space is $\mathbb{R}^n$
② Norm: $\|X\|_1 = |X_1| + |X_2| + \cdots + |X_n|$

Ex in $\mathbb{R}^4$ $X = \begin{bmatrix} -1 \\ 0 \\ 2 \\ -7 \end{bmatrix}$ $\|X\|_1 = 10$

show $\|X\|_1$ is a norm

I $\|V\|_1 \geq 0$ and $\|V\|_1 = 0$ iff $V = 0$
II $\|\alpha V\|_1 = |\alpha| \, \|V\|_1$
III $\|V + W\|_1 \leq \|V\|_1 + \|W\|_1$

check

I $\|V\|_1 = |V_1| + |V_2| + \cdots + |V_n|$
b/c $|V_i| \geq 0$ → $\|V\|_1 \geq 0$ ✓

if $|V_1| + |V_2| + \cdots + |V_n| = 0$
→ $\forall i$ $V_i = 0$ so $V = 0$ ✓

II $\|\alpha V\|_1$
$= |\alpha V_1| + |\alpha V_2| + \cdots + |\alpha V_n|$

$= |\alpha| \left[|V_1| + |V_2| + \cdots + |V_n| \right] = |\alpha| \, \|V\|_1$ ✓

III $\|V + W\|_1 = |V_1 + W_1| + |V_2 + W_2| + \cdots + |V_n + W_n|$
b/c below $\leq \underline{|V_1|} + \underline{\underline{|W_1|}} + \underline{|V_2|} + \underline{\underline{|W_2|}} + \cdots + \underline{|V_n|} + \underline{\underline{|W_n|}}$

Note: for reals and abs value we have

$$|x+y| \le |x| + |y|$$

you can prove by 3 cases

① $x \ge 0, y \ge 0$

② $x < 0, y \ge 0$

③ $x < 0, y < 0$

$$\therefore \quad \|v + w\|_1 \le \|v\|_1 + \|w\|_1$$

5.4 ~~#~~a $\qquad C[-\pi, \pi] \qquad \langle f, g \rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} fg\,dx$

$f = \cos mx$

$g = \sin nx$

① show $f \perp g \xrightarrow{\text{means}}$ show $\langle f, g \rangle = 0$

② show $\|f\| = 1$, $\|g\| = 1 \xrightarrow{\text{means}} \langle f, f \rangle = 1$

$\langle g, g \rangle = 1$

③ $\|f - g\| = \sqrt{\langle f-g, f-g \rangle}$

So ① $\frac{1}{\pi}\int_{-\pi}^{\pi} \cos mx \sin nx\,dx = \cdots = 0$

② $\frac{1}{\pi}\int_{-\pi}^{\pi} \cos^2 mx\,dx = 1$

$\frac{1}{\pi}\int_{-\pi}^{\pi} \sin^2 nx\,dx = 1$

③ $(f-g)^2 = (\cos mx - \sin nx)^2 = \underline{\cos^2 mx} - 2\cos mx \sin nx + \sin^2 nx$

Eigen values and their corresp. eigen vectors

$\boxed{A\mathbb{X} = \lambda \mathbb{X}}$ type problem

Finding $\lambda, \mathbb{X}$

① $\boxed{\det(A - \lambda I)} = 0$

$p(\lambda)$ poly of degree n

Soln's find $\lambda_1, \lambda_2, \ldots, \lambda_n$ eigen values.

③ take $A - \lambda_i I$ for each λ_i

$\rightarrow$ solve $(A - \lambda_i I)\mathbb{X} = \mathbb{0}$ to get the corresp. eigen vectors (eigen space)

(renaming of the null space of $A - \lambda_i I$)

Properties

① if $\lambda = a + bi$ is an eigen value

$\rightarrow \bar{\lambda} = a - bi$ is also one

plus

if $\mathbb{Z}$ is $\lambda = a + bi$ eigen vector

$\rightarrow \bar{\mathbb{Z}}$ is $\bar{\lambda} = a - bi$ eigen vector

ex $\lambda_1 = 3 + 2i$ with $\mathbb{Z} = \begin{bmatrix} 2 \\ 3 - i \end{bmatrix}$

$\rightarrow \lambda_2 = 3 - 2i$ with $\bar{\mathbb{Z}} = \begin{bmatrix} 2 \\ 3 + i \end{bmatrix}$

(2) $\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n$

(3) trace of A ; $tr(A) = a_{11} + a_{22} + \cdots + a_{nn}$

$$tr(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$$

(4) if A is triangular

$\rightarrow \lambda_i = a_{ii}$ (diagonal value are the eigen values)

ex) find λ_i and corresp. X

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$\lambda_1 = 1$
$\lambda_2 = 2$
$\lambda_3 = 4$

λ_1 is eigen vectors

$$X = \begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \left[\begin{array}{ccc|c} 0 & \boxed{2} & 3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \boxed{3} & 0 \end{array}\right]$$

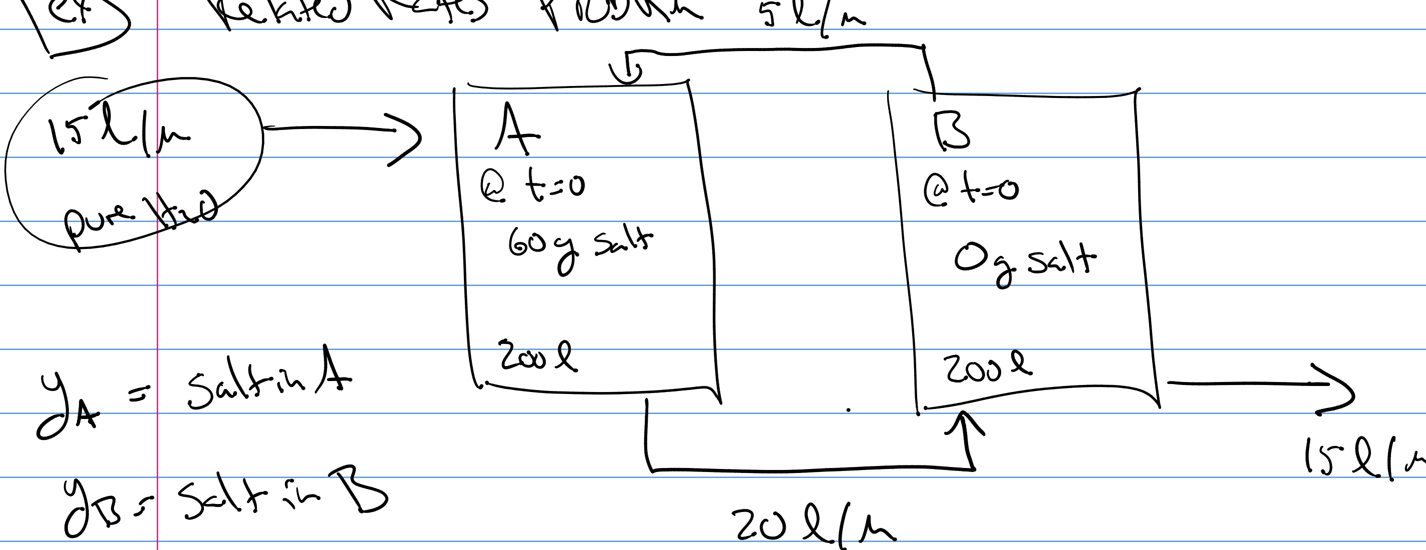
$x_1 = \alpha$
$x_2 = 0$
$x_3 = 0$

$\lambda_1 = 1$ has corresp. vector $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

etc

Why is $A\mathbf{x} = \lambda \mathbf{x}$ interesting?

ex Related Rates Problem



y_A = salt in A

y_B = salt in B

@ $t=0$ $\quad y_A(0) = 60 \quad y_B(0) = 0$

Find $y_A(t) = ? \quad y_B(t) = ?$

$y' = \frac{\text{salt}}{\text{time}}$

Know

$$y'_A = (\text{rate in}) - (\text{rate out})$$

$$y'_A = \left(15\frac{l}{m}\cdot\frac{0g}{l} + 5\frac{\cancel{l}}{min}\cdot\frac{y_B\,g}{200\,\cancel{l}}\right) - \left(20\frac{\cancel{l}}{min}\cdot\frac{y_A\,g}{200\,\cancel{l}}\right)$$

$$y'_A = \left(\frac{1}{40}y_B - \frac{1}{10}y_A\right) = -\frac{1}{10}y_A + \frac{1}{40}y_B$$

$$y'_B = (\text{rate in}) - (\text{rate out})$$

$$y'_B = \left(\frac{1}{10}y_A\right) - \left(5\cdot\frac{y_B}{200} + 15\frac{y_B}{200}\right)$$

$$y'_B = \frac{1}{10}y_A - \frac{1}{10}y_B$$

$$\begin{bmatrix} y'_A = -\frac{1}{10} y_A + \frac{1}{40} y_B \\ y'_B = \frac{1}{10} y_A - \frac{1}{10} y_B \end{bmatrix}$$ Diff Eq's

Systems of Diff. Eq

$\to$ Call $Y = \begin{bmatrix} y_A \\ y_B \end{bmatrix}$ then $Y' = \begin{bmatrix} y'_A \\ y'_B \end{bmatrix}$

$$\Rightarrow \quad Y' = \underbrace{\begin{bmatrix} -1/10 & 1/40 \\ 1/10 & -1/10 \end{bmatrix}}_{\text{call this } A} Y$$

then our sys. of Diff Eq's is $Y' = AY$

Know in 1D for diff eq's

$$\boxed{y' = 3y} \to \frac{dy}{dx} = 3y \to \frac{1}{y} dy = 3dx$$

$$\to \ln y = 3x + C$$

$$\to \quad y = e^{3x+C} = \boxed{c\, e^{3x}}$$

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if 1D had e^x type soln's $\boxed{\text{Guess}}$

exponential soln's for $Y' = AY$

$$Y = \begin{bmatrix} y_A \\ y_B \end{bmatrix} = \begin{bmatrix} x_1 e^{\lambda t} \\ x_2 e^{\lambda t} \end{bmatrix} = e^{\lambda t} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = e^{\lambda t} X$$

check if $Y = e^{\lambda t} X$ is a soln to

$Y' = AY$

$\rightarrow Y' = \lambda e^{\lambda t} X = \lambda Y$

So $Y' = AY \rightarrow \boxed{\lambda Y = AY}$

so if λ is an eigen value of A, X corresp. vector

then $Y = e^{\lambda t} X$ $\boxed{\text{is}}$ a soln!

So Soln to $Y' = \begin{bmatrix} -1/10 & 1/40 \\ 1/10 & -1/10 \end{bmatrix} Y$

is all functions $Y = e^{\lambda t} X = \begin{bmatrix} x_1 e^{\lambda t} \\ x_2 e^{\lambda t} \end{bmatrix}$ $\begin{matrix} // y_A \\ y_B \end{matrix}$

λ is an eigen value of A

w/ X corrs. eigen vector.

So a mixing related rates diff eqn system

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}' = \begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

Sol'n

$$y_i = x_i e^{\lambda t}$$

$\boxed{\text{eigenvalue}}$ λ has $\mathbb{X}$ eigenvector $\mathbb{X} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

Multiple sol'n's

$$y'' + y = 0$$

$y_1 = \sin x \qquad y_2 = \cos x$

Sol'n

$$\boxed{y = a \sin x + b \cos x}$$

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for $Y' = AY \rightarrow \lambda_1, \lambda_2, \ldots, \lambda_n$ eigenvalues of A

Sol'n(s)

$$Y = e^{\lambda_1 t} \mathbb{X}_1$$
$$Y = e^{\lambda_2 t} \mathbb{X}_2$$
$$\vdots$$
$$Y = e^{\lambda_n t} \mathbb{X}_n$$

$\rightarrow Y =$ linear combo of
