Math 511

Q's / Final → Exam 1, Exam 2, Exam 3

remove to get down to 1hr 50 min exam

? ⇒ Modify probs left

⇒ probs do cover concepts.

Eigen Value / Vector solve $Ax = \lambda x$

Application: Solving $Y' = AY$ 1st order system of diff eqn's

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{bmatrix}' = \boxed{A} \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{bmatrix}$$

Solve: $y_1(t) = ?$, $y_2(t) = ?$, ...

Soln's for A find λ_i with eigenvectors x_i

if you have $\boxed{\text{lin. ind. } x_i}$

Soln

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{bmatrix} = Y = \underline{c_1 e^{\lambda_1 t} x_1} + \underline{c_2 e^{\lambda_2 t} x_2} + \dots + \underline{c_n e^{\lambda_n t} x_n}$$

if initial value problem $\rightarrow$ $Y(0) = Y_0$

you can find $c_1 = ?$, $c_2 = ?$, ..., $c_n = ?$ for specific soln.

$\boxed{ex}$ $Y' = AY$ $\qquad Y(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\boxed{\text{Find eigen values + vectors on } A}$

$\hookrightarrow$ $\lambda_1 = 1 \quad X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \lambda_2 = -1 \quad X_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\lambda_3 = 1/2 \quad X_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$Y = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \boxed{c_1 e^{t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{1/2 t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{t} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ c_2 e^{-t} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ c_3 e^{1/2 t} \end{bmatrix}$$

$$\boxed{\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{t} \\ c_2 e^{-t} \\ c_3 e^{1/2 t} \end{bmatrix}}$$

given $Y(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$
$\uparrow$
$t=0$

$$\begin{bmatrix} c_1 e^{0} \\ c_2 e^{-0} \\ c_3 e^{r_2 \cdot 0} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \rightarrow \begin{matrix} c_1 = 1 \\ c_2 = 1 \\ c_3 = 1 \end{matrix}$$

Initial Value soln

$$\boxed{Y = \begin{bmatrix} e^{t} \\ e^{-t} \\ e^{r_2 t} \end{bmatrix}}$$

$Y' = AY$

① Find A's λ_i, X_i ← lin. ind.

② Soln $Y = c_1 e^{\lambda_1 t} X_1 + c_2 e^{\lambda_2 t} X_2 + \cdots + c_n e^{\lambda_n t} X_n$

$\boxed{\text{ex}}$ $Y' = A, Y$

① $\lambda_1 = 1+i \quad X_1 = \begin{bmatrix} i \\ 2+i \end{bmatrix}$ → $Re(X) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ → $Im(X) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2 = 1-i \quad X_2 = \begin{bmatrix} -i \\ 2-i \end{bmatrix}$

② Soln $\boxed{Y = c_1 e^{(1+i)t} \begin{bmatrix} i \\ 2+i \end{bmatrix} + c_2 e^{(1-i)t} \begin{bmatrix} -i \\ 2-i \end{bmatrix}}$

Can we write complex soln's (complex λ's) in a different way?

If A has $\lambda = a + bi$ with corresp. $\mathbb{X}$

$$\lambda = \underbrace{(a)}_{Re(\lambda)} + i\underbrace{(b)}_{Im(\lambda)} \qquad \mathbb{X} = Re(\mathbb{X}) + i\, Im(\mathbb{X})$$

$$\rightarrow \quad \bar{\lambda} = a - bi \qquad \bar{\mathbb{X}} = Re(\mathbb{X}) - i\, Im(\mathbb{X})$$

So we have two soln's

$$Y_1 = e^{\lambda t}\mathbb{X} \qquad Y_2 = e^{\bar{\lambda} t}\bar{\mathbb{X}}$$

$$\rightarrow \quad Y = C_1 e^{\lambda t}\mathbb{X} + C_2 e^{\bar{\lambda} t}\bar{\mathbb{X}} \quad \forall$$

all soln

So any linear combo of Y_1, Y_2 are solns.

Fact: $\frac{1}{2}\big((a + bi) + (a - bi)\big) = a \quad Re(\ )$

$\frac{1}{2}\big((a + bi) - (a - bi)\big) = i\, b \quad Im(\ )$

So $$\frac{1}{2}\left(\boxed{e^{\lambda t}\mathbb{X}} + \boxed{\overline{e^{\lambda t}\mathbb{X}}}\right) = Re\left(e^{\lambda t}\mathbb{X}\right)$$

$$\frac{1}{2}\left(\boxed{e^{\lambda t}\mathbb{X}} - \boxed{\overline{e^{\lambda t}\mathbb{X}}}\right) = Im\left(e^{\lambda t}\mathbb{X}\right)$$

$\boxed{b/c}$ $e^{\lambda t} X = e^{(a+bi)t} X$

$= e^{a}\left((\cos b)t + i(\sin b)t\right)\left(Re(X) + i\,Im(X)\right)$

$Re(e^{\lambda t} X) = Y_1$
$Im(e^{\lambda t} X) = Y_2$ } New Soln's to use as liner combo

let $Y_1 = e^{at}\left(\cos bt\, Re(X) - \sin bt\, Im(X)\right)$

$Y_2 = e^{at}\left(\cos bt\, Im(X) + \sin bt\, Re(X)\right)$

$\lambda = a+bi$ X is λ's eigen value

ex $Y' = A\,Y$

$\lambda_1 = \underset{a=1}{1} + \underset{b=1}{i}$ $X_1 = \begin{bmatrix} i \\ 2+i \end{bmatrix}$ $\to Re(X) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, $Im(X) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\boxed{Sol^n}$

$Y_1 = e^{t}\left(\cos t \begin{bmatrix} 0 \\ 2 \end{bmatrix} - \sin t \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = e^{t}\begin{bmatrix} -\sin t \\ 2\cos t - \sin t \end{bmatrix}$

$Y_2 = e^{t}\left(\cos t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \sin t \begin{bmatrix} 0 \\ 2 \end{bmatrix}\right) = e^{t}\begin{bmatrix} \cos t \\ \cos t + 2\sin t \end{bmatrix}$

$Y = C_1 Y_1 + C_2 Y_2 = C_1 e^{t}\begin{bmatrix} -\sin t \\ 2\cos t - \sin t \end{bmatrix} + C_2 e^{t}\begin{bmatrix} \cos t \\ \cos t + 2\sin t \end{bmatrix}$

till now: $Y' = AY$

New higher order system of diff. eqn's

$$Y'' = A_1 Y + A_2 Y'$$

solve $Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ & find these functions!

$$Y' = \begin{bmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{bmatrix} = \begin{bmatrix} y_{n+1} \\ y_{n+2} \\ \vdots \\ y_{2n} \end{bmatrix}$$

Make a new vector of functions

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \\ y_{n+1} \\ \vdots \\ y_{2n} \end{bmatrix} = \begin{bmatrix} Y \\ Y' \end{bmatrix} = Y_* \quad Y_*' = \begin{bmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \\ y_{n+1}' \\ \vdots \\ y_{2n}' \end{bmatrix} = \begin{bmatrix} Y' \\ Y'' \end{bmatrix}$$

take a partition matrix multiply

$$\begin{bmatrix} Y' \\ Y'' \end{bmatrix} = \left[\begin{array}{c|c} O & I \\ \hline A_1 & A_2 \end{array}\right] \begin{bmatrix} Y \\ Y' \end{bmatrix}$$

becomes:

$$\begin{bmatrix} Y' \\ Y'' \end{bmatrix} = \begin{bmatrix} OY + IY' \\ A_1 Y + A_2 Y' \end{bmatrix} = \begin{bmatrix} Y' \\ A_1 Y + A_2 Y' \end{bmatrix}$$

to solve $Y'' = A_1 Y + A_2 Y'$ $(n \times n)$

is to solve $(2n \times 2n$ system$)$

$$Y_*' = \begin{bmatrix} 0 & I \\ A_1 & A_2 \end{bmatrix} Y_*$$

where $Y_* = \begin{bmatrix} Y \\ Y' \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \\ y_{n+1} \\ \vdots \\ y_{2n} \end{bmatrix}$ $\left.\begin{matrix} y_{n+1} \\ \vdots \\ y_{2n} \end{matrix}\right\} \begin{matrix} = y_1' \\ \vdots \\ = y_n' \end{matrix}$

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ex $Y'' = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} Y + \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} Y'$

solve: $Y_*' = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 2 & -1 \\ 0 & -1 & -1 & 3 \end{bmatrix}}_{= A} Y_*$

Soln: find A's eigen values, vectors

λ_1, x_1 $\quad \lambda_2, x_2$ $\quad \lambda_3, x_3$ $\quad \lambda_4, x_4$

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>> [c d] = eig(A)
c =
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$$\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \\ -0.156988 & -0.255039 & -0.876877 & -0.384907 \\ 0.237950 & -0.407082 & -0.160157 & 0.871074 \\ -0.527847 & -0.465647 & 0.445875 & -0.123308 \\ 0.800070 & -0.743244 & 0.081437 & 0.279055 \end{array}$$

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d =

Diagonal Matrix
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$$\begin{array}{cccc} \overset{\lambda_1}{3.36234} & 0 & 0 & 0 \\ 0 & \overset{\lambda_2}{1.82578} & 0 & 0 \\ 0 & 0 & \overset{\lambda_3}{-0.50848} & 0 \\ 0 & 0 & 0 & \overset{\lambda_4}{0.32036} \end{array}$$

Now make linear combo