Math 511

Q's/

ex $A = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$ find eigenvalues

$\lambda = 1, 2, 3$

ex $A = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & -1 & 3 \\ 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$ find eigenvalues

$$\begin{vmatrix} 1-\lambda & 2 & 0 & 4 \\ 0 & 1-\lambda & -1 & 3 \\ 1 & 0 & 2-\lambda & 1 \\ 0 & 0 & 1 & 3-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 1-\lambda & -1 & 3 \\ 0 & 2-\lambda & 1 \\ 0 & 1 & 3-\lambda \end{vmatrix} + (1)\begin{vmatrix} 2 & 0 & 4 \\ 1-\lambda & -1 & 3 \\ 0 & 1 & 3-\lambda \end{vmatrix}$$

$$= \dots = p(\lambda)$$

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ex $A = \begin{bmatrix} 3 & -8 \\ 2 & 3 \end{bmatrix}$

$$\begin{vmatrix} 3-\lambda & -8 \\ 2 & 3-\lambda \end{vmatrix} = (3-\lambda)^2 + 16$$

$$= 9 - 6\lambda + \lambda^2 + 16 = \lambda^2 - 6\lambda + 25$$

Solve: $\lambda^2 - 6\lambda + 25 = 0$

$$\lambda = \frac{6 \pm \sqrt{36 - 100}}{2} = \frac{6 \pm \sqrt{-64}}{2}$$

$$\lambda = \frac{6 \pm 8i}{2} = 3 \pm 4i$$

$$A = \begin{bmatrix} 3 & -8 \\ 2 & 3 \end{bmatrix}$$

$$\lambda_1 = 3+4i \rightarrow \left[\begin{array}{cc|c} 3-(3+4i) & -8 & 0 \\ 2 & 3-(3+4i) & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cc|c} -4i & -8 & 0 \\ 2 & -4i & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 4 & -8i & 0 \\ 2 & -4i & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cc|c} 1 & -2i & 0 \\ 1 & -2i & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} \boxed{1} & -2i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_2 = \alpha$

$$X_1 = \begin{bmatrix} 2\alpha i \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} 2i \\ 1 \end{bmatrix} \rightarrow x_1 = 2\alpha i$$

$$x_1 = \alpha \begin{bmatrix} 2i \\ 1 \end{bmatrix} \to \begin{bmatrix} 2i \\ 1 \end{bmatrix} \quad \alpha = 1$$

$$\to \begin{bmatrix} -2 \\ i \end{bmatrix} \quad \alpha = i$$

$$\boxed{\begin{array}{ll} \lambda_1 = 3+4i & x_1 = \begin{bmatrix} 2i \\ 1 \end{bmatrix} \\ \lambda_2 = \bar{\lambda}_1 = 3-4i & x_2 = \bar{x}_1 = \begin{bmatrix} -2i \\ 1 \end{bmatrix} \end{array}}$$

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eigen

$A \to \lambda_i, x_i$ eigenvalue / vector

$$AX = XD$$

$X = [x_1 \ x_2 \ \dots \ x_n]$

$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

if X^{-1} ← x_i must be <u>lin. ind.</u>

$$\boxed{A = X D X^{-1}}$$ factorization

or

$$\boxed{X^{-1} A X = D}$$ diagonalization

Also $A_{n \times n}$ we can think of it as a transformation (function)

$$\mathbb{R}^n \xrightarrow{A} \mathbb{R}^n$$

$\mathbb{R}^n$ standard basis is $\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix} \cdots$

$e_1, e_2, \ldots, e_n$

$\dim(\mathbb{R}^n) = n$ so $\boxed{\text{any}}$ n linearly ind. vectors act as a $\boxed{\text{basis}}$

back to $A = XDX^{-1}$

$X = \underline{\underline{[x_1, \ldots, x_n]}}$ n - linearly ind. eigen vectors

so X is <u>also</u> a basis of $\mathbb{R}^n$

<u>Change of Basis</u> $B = [b_1, b_2 \ldots b_n]$ is a basis

$$[c]_E = B[c]_B$$

$$[c]_B = B^{-1}[c]_E$$

$\boxed{\text{Now}}$ L is a transform on $\mathbb{R}^n$

+ $\boxed{A}$ is the matrix rep. of L in standard basis.

$L(x_E) = y_E$ $\boxed{A x_E = y_E}$

Now: $A = X D X^{-1}$

$$\boxed{A [\vec{x}]_E} = X \boxed{D} X^{-1} [\vec{x}]_E$$

$[\ ]_E$? $[\vec{x}]_{X \leftarrow \text{eigen vectors}}$

$$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

A is L's rep. is standard

D is L's rep is in eigen vector basis

Stochastic Process : any seq for which the next outcome depends on chance

Markov Process : Stochastic process such that

1. States are finite
2. next is function of previous
3. probabilities are const. over iterations

$\boxed{\text{Markov Chain}}$ ① $\vec{x}_0, \vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k, \ldots$: state vectors

② A : transition matrix $A \vec{x}_{k-1} = \vec{x}_k$

(3) any v is a probability vector if values are non-neg and sum to one.

ex $v = \begin{bmatrix} .01 \\ .90 \\ .09 \end{bmatrix}$

(4) $A = [a_i]$ if a_i are prob. vectors

$\rightarrow$ A is a stochastic matrix

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p. 145 population migration.

$A x_0 = x_1$

ex $x_0 = \begin{bmatrix} .5 \\ .5 \end{bmatrix}$

$$A = \begin{bmatrix} .94 & .02 \\ .06 & .98 \end{bmatrix}$$

$\lambda_1 = .92 \qquad x_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\lambda_2 = 1 \qquad x_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$$A = X \begin{bmatrix} .92 & 0 \\ 0 & 1 \end{bmatrix} X^{-1}$$

$$v_{k+1} = A v_k = X \begin{bmatrix} .92 & 0 \\ 0 & 1 \end{bmatrix} X^{-1} [v_k]_E$$

$$v_k = A^k v_0 = X \begin{bmatrix} \boxed{.92^k}^{\to 0} & 0 \\ 0 & \boxed{1^k} \end{bmatrix} X^{-1} [v_0]_E$$

$= \mathbb{X}_1 \cdot 0 + \boxed{\mathbb{X}_2 \cdot \boxed{\#}}$

Thm: for $A_{n\times n}$. If the markov chain converges to a steady-state vector $\mathbb{X}$

then ① $\mathbb{X}$ is prob. vector

② $\lambda_1 = 1$ is an eigen value of A

and $\mathbb{X}$ is an eigen vector belonging to $\lambda_1 = 1$

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another application: exponentiation

$a \in \mathbb{R}$: $\boxed{e^a} = 1 + a + \frac{1}{2!}a^2 + \frac{1}{3!}a^3 + \dots = \boxed{\sum_{k=0}^{\infty} \frac{1}{k!} a^k}$

Idea: $e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \dots$

$A_{n\times n}$

$$\boxed{e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k}$$

if $A = XDX^{-1}$

$A^k = X \boxed{D^k} X^{-1}$

try $I + D + \frac{1}{2!}D^2 + \frac{1}{3!}D^3 + \dots$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} D^k \qquad D^k = \begin{bmatrix} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \lambda_n^k \end{bmatrix}$$

$$= \sum_{k=0}^{\infty} \begin{bmatrix} \frac{1}{k!}\lambda_1^k & & & 0 \\ & \frac{1}{k!}\lambda_2^k & & \\ & & \ddots & \\ 0 & & & \frac{1}{k!}\lambda_n^k \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{k=0}^{\infty} \frac{1}{k!}\lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \sum_{k=0}^{\infty} \frac{1}{k!}\lambda_n^k \end{bmatrix}$$

$$\boxed{e^D = \begin{bmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{bmatrix}}$$

but $A = XDX^{-1}$

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k = \sum_{k=0}^{\infty} \underline{\frac{1}{k!}} \left[\boxed{X} D^k \boxed{X^{-1}} \right]$$

$$e^{A} = X\left[\sum_{k=0}^{\infty} \frac{1}{k!} D^{k}\right] X^{-1}$$

$$e^{A} = X e^{D} X^{-1}$$

so $$\boxed{e^{A} = X \begin{bmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{bmatrix} X^{-1}}$$

Why?!

$$Y' = AY, \quad Y(0) = Y_0$$

guess: $Y = e^{tA}$

Check

$$Y' = \frac{d}{dt}\left[e^{tA}\right] = \frac{d}{dt}\left[I + tA + \frac{1}{2!}t^2 A^2 + \frac{1}{3!}t^3 A^3 + \dots\right]$$

$$= \left[\emptyset + A + tA^2 + \frac{1}{2!}t^2 A^3 + \dots\right]$$

$$= A\left[I + tA + \frac{1}{2!}t^2 A^2 + \dots\right]$$

$$= AY$$

so $Y = e^{tA}$ is a soln!

Notice: $Y(0) = e^{0 \cdot A} = I$

$\bigstar$ let $\boxed{Y = e^{tA} Y_0}$ soln to $Y' = AY$ and $Y(0) = Y_0$

$\Rightarrow Y(0) = Y_0$

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ex 6.2 $A = \begin{bmatrix} -1/10 & 1/40 \\ 1/10 & -1/10 \end{bmatrix}$ $Y_0 = \begin{bmatrix} 60 \\ 0 \end{bmatrix}$

$Y' = AY$, $Y(0) = Y_0$

$$\boxed{Y = e^{t\begin{bmatrix} -1/10 & 1/40 \\ 1/10 & -1/10 \end{bmatrix}} \begin{bmatrix} 60 \\ 0 \end{bmatrix}}$$

$\to$ remember e^{tA}

$$= X \begin{bmatrix} e^{t\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{t\lambda_n} \end{bmatrix} X^{-1}$$