Math 511 - Final Exam Review

Exam 1 Problems

1) Solve the system of equations. Do not use matrices.

$$\begin{aligned} x + y + u &= 4 \\ 2y + u &= 4 \\ x - 2y + z + u &= 2 \\ 2x + y + z - u &= 2 \end{aligned}$$

2) Solve the system of equations. Use Gaussian Elimination on an augmented matrix.

$$\begin{aligned} x + y + u &= 4 \\ 2y + u &= 4 \\ x - 2y + z + u &= 2 \\ 2x + y + z - u &= 2 \end{aligned}$$

3) Solve the system of equations by using Gauss-Jordan elimination on an augmented matrix.

$$\begin{aligned} x - y &= 16 \\ y - z &= -4 \\ z - u &= 21 \\ u - x &= -33 \end{aligned}$$

4) Perform the indicated operations.

a) $\begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}^T - 3 \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}^2$

b) Perform and name this operation: $\begin{pmatrix} 1 & -2 & 3 \end{pmatrix} \begin{pmatrix} 1 & -2 & 3 \end{pmatrix}^T$

c) Perform and name this operation: $\begin{pmatrix} 1 & -2 & 3 \end{pmatrix}^T \begin{pmatrix} 1 & -2 & 3 \end{pmatrix}$

5) Let

$$A = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}, B = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}, \text{and } C = \begin{pmatrix} 3 & 0 \\ 2 & 1 \end{pmatrix}$$

Solve $X - 3A^2 + B = C^T + 2B$ for matrix X.

6) Find the LU factorization for the the given matrix.

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 4 & 3 & 1 \\ 2 & 4 & 4 \end{pmatrix}$$

7) Find A^{-1} for the the given matrix (Hint: continue your work from above because it is the same matrix A).

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 4 & 3 & 1 \\ 2 & 4 & 4 \end{pmatrix}$$

8a) State Theorem 1.5.2

8b) Let $A\boldsymbol{x} = \boldsymbol{b}$ be a system of n linear equations with n unknowns, and suppose that $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are both solutions and $\boldsymbol{x}_1 \neq \boldsymbol{x}_2$. Is the matrix A singular or non-singular? How many solutions will the system have? Explain. (Hint: Use the fact that $A\boldsymbol{x}_1 - A\boldsymbol{x}_2 = \boldsymbol{0}$)

9) Let

$$A = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}, B = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}, \text{and } C = \begin{pmatrix} 3 & 0 \\ 2 & 5 \end{pmatrix}$$

Solve $XA - B = C$ for matrix X.

10) Let A and B be 10×10 matrices that are partitioned into sub-matrices as follows:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

If A_{11} is a 6×5 matrix and B_{11} is a $k \times r$ matrix, what conditions, if any, must k and r satisfy in order to make the block multiplication of A and B possible?

11) Given matrix A

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 0 & 0 \\ 4 & 3 & 1 & 1 \\ 1 & 1 & 0 & 2 \end{pmatrix}$$

a) Find $\det(A)$ by co-factors.

b) Does A have an inverse? Explain.

12) Given matrix A

$$A = \begin{pmatrix} 1 & x_1 & 1 \\ 1 & x_2 & 2 \\ 1 & x_3 & 3 \end{pmatrix}$$

Find the determinate by using elimination. What conditions must the three scalars x_i satisfy for A to be non-singular?

Exam 2 Problems

1) For the set of vectors in $\mathbb{R}^2$ define addition normally but scalar multiplication by $\alpha \boldsymbol{x} = [\alpha x_1, x_2]^T$. Does this form a vector space? Explain. (Note: Axioms are given on the last page of the exam)

2a) Is the set of all polynomials in P_4 of odd degree a subspace of P_4? Explain.

2b) Let A be an $m \times n$ matrix. Show that $N(A)$ is a subspace of $\mathbb{R}^n$.

check only closure

3) Show that $\{(1), (e^x + e^{-x}), (e^x - e^{-x})\}$ are linearly independent in $C[0,1]$.

4) Determine whether $[1,1,3]^T$ and $[0,2,1]^T$ are linearly independent in $\mathbb{R}^3$.

5) Consider the vectors $\boldsymbol{x}_1 = [1,2]^T, \boldsymbol{x}_2 = [2,5]^T, \boldsymbol{x}_3 = [3,7]^T$, and $\boldsymbol{x}_4 = [3,4]^T$. Pare down or extend the vectors to make a basis for $\mathbb{R}^2$.

6) For a vector space V with bases B_1 and B_2, find the transition matrix, S, representing the change of base from B_2 to B_1. Find the transition matrix, T, representing the change of base from B_1 to B_2. Explain the action of multiplying $S[\boldsymbol{x}]_{B_2}$ and $T[\boldsymbol{x}]_{B_1}$.

7) Determine if $L(\boldsymbol{x}) = [x_2, x_1, x_1 + x_2^2]^T$ from $\mathbb{R}^2$ to $\mathbb{R}^3$ is a linear transform.

8) Let A be a 3×4 matrix and U is the reduced row echelon form of A. If ...

$$U = \begin{pmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

a) determine the rank(A), the nullity of A, and the dependency equations.

b) if $\boldsymbol{a}_1 = [1,2,3]^T, \boldsymbol{a}_2 = [1,1,1]^T$ find $\boldsymbol{a}_3$ and $\boldsymbol{a}_4$.

c) find a basis for the column space of A and a basis for the row space of A.

d) find a basis for N(A).

9) Determine the kernel and range of the linear transform $L(p(x)) = p'(x)$ on P_3.

10) For the linear transformation $L(\boldsymbol{x}) = [x_1 + x_2, x_2 + x_3]^T$ from $\mathbb{R}^3$ to $\mathbb{R}^2$ find the standard linear transformation matrix, A.

11) For the linear transformation $L(\boldsymbol{p}) = x\boldsymbol{p}' + \boldsymbol{p}''$ on P_3. Find the matrix, A, representing L with respect to the standard basis. And find the matrix, T, representing L with respect to the ordered basis $[(1), (1 + x), (x + x^2)]$.

Exam 3 Problems

1) For the vectors $\boldsymbol{x} = (4, 3, 2)^T$ and $\boldsymbol{y} = (1, 1, 1)^T$, find the vector projection $\boldsymbol{p}$ of $\boldsymbol{x}$ onto $\boldsymbol{y}$, and verify that $(\boldsymbol{x} - \boldsymbol{p}) \perp \boldsymbol{p}$.

2) Let A be a 4x3 matrix. Considering A as a linear transform describe its domain, codomain, and the relationships between $N(A)$, $N(A^T)$, $R(A)$, and $R(A^T)$. For its domain, is it possible for A to have the vector $(3, 1, 2)$ in its row space and $(-1, 1, 1)^T$ in its null space? Explain.

3) For an experiment you collect the following (x, y)-data points: $\{(0, 0), (1, 1), (2, 0)\}$. Solve the least-squares fit to the data by a line.

4) Given inner product space $C[-1, 1]$ with $< f, g >= \int_{-1}^{1} f(x)g(x)dx$, find the projection of $f(x) = x^2 + 1$ onto $g(x) = x + 1$.

5) Given inner product space $\mathbb{R}^{2\times 2}$ with $\langle A, B\rangle = a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + a_{22}b_{22}$

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$$

for an inner product space $\cos(\theta) = \frac{\langle \boldsymbol{u}, \boldsymbol{v}\rangle}{||\boldsymbol{u}||||\boldsymbol{v}||}$, find the angle between A and B. (Note: leave your answer in arccos form)

6) Do the functions $\sqrt{3}\ x$ and $\sqrt{5}\ x^2$ form an orthonormal set in $C[-1,1]$ with the inner product defined by $< f, g >= \frac{1}{2}\int_{-1}^{1} f(x)g(x)dx$? (Explain)

7) Use that fact that the functions $\cos(2x)$ and $\sin(x)$ form an orthonormal set in $C[-\pi,\pi]$ with the inner product defined by $< f, g >= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)dx$. Determine the value of ...

$\int_{-\pi}^{\pi}(2\sin(x) - 3\cos(2x))(2\cos(2x) + 3\sin(x))dx.$

8) Use the Gram-Schmidt process to find the QR factorization of $A = \begin{pmatrix} 2 & 6 \\ 1 & 10 \\ 1 & -3 \end{pmatrix}$

9) Find the eigenvalues and corresponding eigenvectors for the matrix $A = \begin{pmatrix} 3 & 2 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

10) Find the eigenvalues and corresponding eigenvectors of the matrix $A = \begin{pmatrix} 3 & -8 \\ 2 & 3 \end{pmatrix}$

11) Reading a word problem you work out that the system of differential equations with initial values is given by:

$$\begin{aligned} y'_1 &= -y_1 - y_2 + y_3 \\ y'_2 &= -2y_2 - y_3 \\ y'_3 &= -3y_3 \end{aligned}$$

with initial values of $y_1(0) = 1$, $y_2(0) = 1$, and $y_3(0) = 1$. Solve the system.

2

12) For the given matrix A, find the diagonal factorization of A.

$$A = \begin{pmatrix} 9 & -5 & 3 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$